Discovering the mathematics of quantum Rabi models

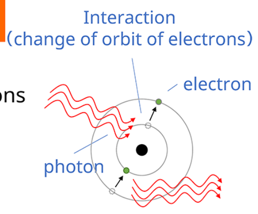
Zeta functions in the interaction of light and matter

Abstract

The study of the interaction between light and matter is fundamental for high impact applications in quantum information science including the realization of quantum computers. In this research, to understand the spectral zeta function of the quantum Rabi model (QRM), one of the most basic models describing interaction of light and matter, we devised a completely new method for the computation of explicit formula of the heat kernel of the QRM. This new method avoids the use of (continuous) path integrals by using a combination of techniques from several areas of mathematics, including combinatorics and representation theory. In addition to the study of the spectral zeta function and its related arithmetics, the resulting formula gives new insights into the physics of quantum interaction and is expected to have many new applications.

Quantum Rabi model(QRM)

- A model that described the **interaction** between **light and matter** with applications to quantum information science.
- Theoretical results using the QRM agree with experiments (Braak,2011). Since exact computations of the QRM are very difficult, conventional research is limited to approximations.



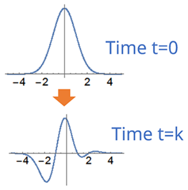
Light and matter interaction

Light energy — Matter (atom) energy — Energy of interaction

$$H_R = \omega a^\dagger a + \Delta \sigma_z + g(a + a^\dagger)\sigma_x$$

Definition of QRM (Hamiltonian)

Time evolution and heat kernel



Time evolution of quantum states

Result ①

Explicit computation of **heat kernel**, a function used to control the **time evolution** of the QRM

Usual method

Path integral using "infinite number of continuous paths"

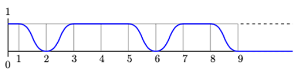
➡ Extremely difficult to obtain rigorous results

(source：Wikipedia)

Proposed method

Computation using "**discrete paths**"

➡ Enables the explicit computation



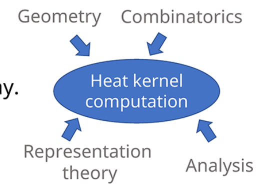
Application

- Explicit form of heat kernel K_R ➡ Enables analysis of physical properties[1][2]

$$K_R(x,y,t) = K(x,y,t;g) \sum_{\lambda=0}^{\infty} (t\Delta)^\lambda \, \Phi_\lambda(x,y,t;g)$$

t: time — Heat kernel of Quantum harmonic oscillator — Matrix expression obtained by new method

- Our method combines techniques from several areas of mathematics in a new way.
  ➡ This method may be extended for the computation of heat kernel of models other than the QRM.



Relation with Zeta functions

Relation between zeta function and heat kernel

The study of the heat kernel is related to mysterious formulas

$$\star:\ 1+1+1+1\cdots \text{“=”} -\tfrac{1}{2} \qquad ☼:\ 1+2+3+4\cdots \text{“=”} -\tfrac{1}{12}$$

Non-rigorous formulas giving values to divergent series found in 18th century

The values of these formulas may be obtained by the analytic continuation of the **Riemann zeta function**. This function contains the information of the prime numbers, and it is one of the most important functions in mathematics.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1 \qquad \star: \zeta(0) = -\tfrac{1}{2} \qquad ☼: \zeta(-1) = -\tfrac{1}{12}$$

Riemann zeta function — Values obtained by analytic continuation

Using the energy levels ($\lambda_1, \lambda_2, \lambda_3 \cdots$) of the QRM we define the **spectral zeta function** of the QRM as an analog of the Riemann zeta function.

$$\zeta_R(s;\tau) = \sum_{n=1}^{\infty} \frac{1}{(\lambda_n + \tau)^s}, \qquad \Re(s) > 1$$

Spectral zeta function

The investigation of the properties of this function was the starting point of this research.

Result ②

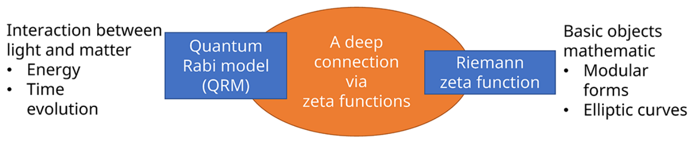
The heat kernel allows to express the spectral zeta function with an **integral expression** and give its analytic continuation using the properties of the heat kernel [2] (Sugiyama,2018).

$$\zeta_R(s;\tau) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{0+} (-\omega)^{s-1} e^{-\tau\omega}\, Tr\left(\int_{-\infty}^{\infty} K_R(\omega,\omega,t)dt\right) d\omega \qquad s \in \mathbb{C}$$

Integral expression of the spectral zeta function

Future research

There is a connection between **special values** of zeta function (like $\zeta(3)$) and values of spectral zeta functions. In the future we would like to study and clarify these connections.



Towards the future of mathematics via the study of zeta functions

References

[1] C. Reyes-Bustos, M. Wakayama, "The heat kernel for the quantum Rabi model", *Adv. Theor. Math. Phys.*, Vol. 26, No. 5, pp. 1347-1447, 2022.

[2] C. Reyes-Bustos, M. Wakayama, "Heat kernel for the quantum Rabi model: II. Propagators and spectral determinants", *J. Phys. A: Math. Theor.* , Vol. 54, 115202, 2021.

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